Journal of Statistical Physics, Vol. 16, No. 4, 1977

An Intermediate Spherical Model of a Ferromagnet[1]

Wayne W. Barrett[2]

Received December 2, 1976

A one-parameter family of partition functions is considered which for zero value of the parameter α reduces to the spherical model of a ferromagnet. The model for $\alpha > 0$ is closer to the usual discrete lattice spin model of a ferromagnet than is the spherical model. The first four terms in α of the limiting value of the partition function are calculated above and below the critical temperature for arbitrary interactions using the saddle point method to calculate certain correlation functions for the spherical model. These calculations indicate that the critical temperature is independent of α for small α and certain interactions.

KEY WORDS: Phase transition; spherical model; saddle point method.

1. INTRODUCTION

One of the outstanding problems in theoretical physics is understanding phase transitions. For a ferromagnet, such as iron, the problem is to explain the fact that when it is placed in an external magnetic field it becomes magnetized, but when the field is removed, then above a certain critical temperature T_c the magnet loses its magnetism but below T_c it retains it. The key to the solution of this problem is the evaluation of the partition function for the system. For a now commonly considered lattice spin model of a ferromagnet, the partition function is

$$Q_N(\nu, h) = \sum_{\{\mu\}} \exp\left(\frac{\nu}{2} \sum_{i,j=1}^{N} \rho_{ij}\mu_i\mu_j + h \sum_{1}^{N} \mu_i\right)$$

where N is the number of sites in the lattice; $\nu = J(kT)^{-1}$, where $J > 0$ is a magnetization constant, k is Boltzmann's constant, and T is the absolute temperature; h is the external magnetic field; $\rho_{ij} = \rho(|\mathbf{r}_i - \mathbf{r}_j|) \geq 0$ is the

---

[1] Part of this research appeared in the author's doctoral thesis.[3]

[2] Department of Mathematics, University of Wisconsin—Madison, Madison, Wisconsin.

interaction between two lattice sites $\mathbf{r}_i$ and $\mathbf{r}_j$ in space; μ_i is a spin variable assuming the values ± 1; and $\sum_{\{\mu\}}$ denotes the sum over the 2^N possible spin configurations $\mu = (\mu_1, \ldots, \mu_N)$.

While $Q_N(\nu, h)$ is jointly analytic in ν and h for all N, if we take the so-called thermodynamic limit

$$-\psi/kT = \lim_{N\to\infty} (1/N) \log Q_N(\nu, h)$$

where ψ is the free energy per site, we may expect nonanalyticities, particularly on the line $h = 0$. A phase transition point is defined to be any nonanalytic point of this limit.

The evaluation of the partition function is difficult and there are just a few cases for which it has been evaluated exactly, corresponding to special choices for the interaction ρ. For example, in this notation, the Curie–Weiss model corresponds to $\rho_{ij} \equiv 1/N$ and the Ising model to $\rho_{ij} = 1$ if $|\mathbf{r}_i - \mathbf{r}_j| = 1$ and 0 otherwise. As is known from the Ising model, the difficulty in evaluating the partition function increases immensely with the space dimension.

In 1952 Kac and Berlin (1) introduced a mathematical variation of a lattice spin model called the spherical model. In it the sum over configurations $\sum_{\{\mu\}}$, which is the sum over the vertices of a cube in N dimensions, is replaced by the integral over a sphere passing through these vertices. Thus the discrete variables $(\mu_1, \ldots, \mu_N)$, $\mu_i = \pm 1$, $i = 1, \ldots, N$, which satisfy the condition

$$\sum_{i=1}^{N} \mu_i^2 = N$$

are replaced by continuous variables $(x_1, \ldots, x_N)$, where the x_i are constrained to lie on the N-dimensional sphere with radius $\sqrt{N}$:

$$\sum_{i=1}^{N} x_i^2 = N$$

With this modification the partition function $Q_N(\nu, h)$ becomes

$$Q_N(\nu, h) = \int_{\sum_1^N x_k^2 = N} \exp\left(\frac{\nu}{2} \sum_{i,j=1}^{N} \rho_{ij} x_i x_j + h \sum_{1}^{N} x_i\right) d\sigma_{\sqrt{N}}$$

where $d\sigma_{\sqrt{N}}$ represents the surface element of the N-dimensional sphere with radius $\sqrt{N}$.

By using the saddle point method, Kac and Berlin evaluated this partition function for the Ising model in one, two, and three dimensions by a method essentially independent of dimension. They found that there is no phase transition in one and two dimensions, but in three dimensions there is one.

Recently Kac suggested investigating the following modified spherical model. The last partition function is replaced by

$$Q_N(\nu, h, \alpha) = \int_{\sum_1^N x_k{}^2 = N} \left[\exp\left(\frac{\nu}{2} \sum_{i,j=1}^{N} \rho_{ij} x_i x_j + h \sum_1^N x_i\right)\right] \prod_1^N (1 + \alpha x_k{}^2)\, d\sigma_{\sqrt{N}} \tag{1}$$

where the weight function $\prod_1^N (1 + \alpha x_k{}^2)$ has been introduced. Here α is a positive real number. This weight function has its maxima at the 2^N points $(\pm 1, \pm 1, ..., \pm 1)$, which are the points μ_i that were summed over in the original discrete partition function. Thus it will be closer to that partition function but can be investigated by similar techniques to those used in evaluating the spherical model partition function.

Mathematically, the problem is to evaluate the thermodynamic limit of the partition function

$$q(\nu, h, \alpha) = \lim_{N \to \infty} (1/N) \log Q_N(\nu, h, \alpha) \tag{2}$$

We have conjectured (2) for $h = 0$ that if the spherical model ($\alpha = 0$) already exhibits a phase transition, then for sufficiently small $\alpha > 0$ the modified spherical model also exhibits a phase transition, *and furthermore* that the phase transition point ν_c where the limit is nonanalytic *is independent of* α, at least for α sufficiently small. In this paper the limit for general ρ_{ij} is expressed as a power series in α and the first few terms are calculated explicitly. The results agree with the above conjecture.

A difficulty occurs in calculating

$$\lim_{N \to \infty} (1/N) \log Q_N(\nu, 0, \alpha)$$

directly for $\nu \geqslant \nu_c$. We have calculated

$$\lim_{h \to 0} \lim_{N \to \infty} (1/N) \log Q_N(\nu, h, \alpha)$$

i.e., we first find the limit for nonzero h and then take the limit as $h \to 0$. The result is the same because $q(\nu, h, \alpha)$ is continuous at $h = 0$.

Much intuition can be gained on the problem of calculating the limit (2) by evaluating it explicitly for the cases $\rho_{ij} \equiv 0$ and $\rho_{ij} \equiv 1/N$. However, we will not go into this here and refer the reader to Refs. 2 and 3.

2. FORMAL EXPANSION OF THE PARTITION FUNCTION IN POWERS OF α

Since it does not appear that the partition function of the modified spherical model is exactly soluble for general ρ_{ij}, we now present a method of calculating the partition function term by term as a power series in α. Expanding $\prod_1^N (1 + \alpha x_k^2)$ in (1), we can write

$$\frac{Q_N(\nu, h, \alpha)}{Q_N(\nu, h, 0)} = 1 + \alpha \sum_{i=1}^N \langle x_i^2 \rangle + \frac{\alpha^2}{2!} \sum_{i,j=1}^N \langle x_i^2 x_j^2 \rangle (1 - \delta_{ij})$$
$$+ \frac{\alpha^3}{3!} \sum_{i,j,k=1}^N \langle x_i^2 x_j^2 x_k^2 \rangle (1 - \delta_{ij})(1 - \delta_{ik})(1 - \delta_{jk}) + \cdots$$

where $\langle\ \rangle$ are the "spherical averages" (B.10) and δ is the usual Kronecker delta. One now introduces cluster functions (see Ref. 2 or Ref. 3). Set

$$\chi_1(i) = \langle x_i^2 \rangle$$

and define the successive χ's by the formulas

$$\langle x_i^2 x_j^2 \rangle (1 - \delta_{ij}) = \chi_1(i)\chi_1(j) + \chi_2(i, j)$$
$$\langle x_i^2 x_j^2 x_k^2 \rangle (1 - \delta_{ij})(1 - \delta_{ik})(1 - \delta_{jk}) = \chi_1(i)\chi_1(j)\chi_1(k) + \chi_1(i)\chi_2(j, k) + \chi_1(j)\chi_2(i, k) + \chi_1(k)\chi_2(i, j) + \chi_3(i, j, k)$$
$$\vdots$$

One then has a *rigorous* identity

$$\frac{Q_N(\nu, h, \alpha)}{Q_N(\nu, h, 0)} = \exp\left[\sum_{k=1}^{\infty} \frac{\alpha^k}{k!} \sum_{i_1, i_2, \ldots, i_k = 1}^N \chi_k(i_1, i_2, \ldots, i_k)\right] \tag{3}$$

The first term is given by

$$\sum_{i=1}^N \chi_1(i) = \sum_{i=1}^N \langle x_i^2 \rangle = \left\langle \sum_{i=1}^N x_i^2 \right\rangle = N$$

since the integration is over the sphere $\sum_1^N x_k^2 = N$. By considering only periodic lattices (in one dimension, sites on a ring; in two dimensions, sites on a torus; etc.), every site looks the same and $\langle x_i^m \rangle = \langle x_j^m \rangle$ for all i, j, m. Then elementary calculations give

$$\sum_{i,j=1}^N \chi_2(i, j) = -N\langle x_1^4 \rangle \qquad \text{and} \qquad \sum_{i,j,k=1}^N \chi_3(i, j, k) = 2N\langle x_1^6 \rangle$$

Thus

$$\frac{1}{N} \log \frac{Q_N(\nu, h, \alpha)}{Q_N(\nu, h, 0)} = \alpha - \frac{\alpha^2}{2} \langle x_1^4 \rangle + \frac{\alpha^3}{3} \langle x_1^6 \rangle - \cdots$$

to the first three terms.

This method can be continued but the succeeding terms are considerably more involved. The following method is more efficient for the purposes of calculation.

Let $\phi(x) = \prod_1^N (1 + \alpha x_k^2)$ and write

$$\frac{1}{N}\log\frac{Q_N(\nu, h, \alpha)}{Q_N(\nu, h, 0)} = \frac{1}{N}\log\langle\phi(x)\rangle = \frac{1}{N}\log\langle e^{\log\phi(x)}\rangle$$

We use the cumulant expansion:

$$\begin{aligned}\log\langle e^{\log\phi(x)}\rangle = \langle\log\phi(x)\rangle &+ \frac{1}{2}\langle[\log\phi(x) - \langle\log\phi(x)\rangle]^2\rangle \\ &+ \frac{1}{3!}\langle[\log\phi(x) - \langle\log\phi(x)\rangle]^3\rangle \\ &+ \frac{1}{4!}\{\langle[\log\phi(x) - \langle\log\phi(x)\rangle]^4\rangle \\ &- 3\langle[\log\phi(x) - \langle\log\phi(x)\rangle]^2\rangle^2\} + \cdots\end{aligned}$$

Then, using $\log\phi(x) = \sum_1^N \log(1 + \alpha x_k^2)$ and the expansion

$$\log(1 + \alpha x_k^2) = \alpha x_k^2 - \tfrac{1}{2}\alpha^2 x_k^4 + \tfrac{1}{3}\alpha^3 x_k^6 - \cdots$$

we can simplify the above expression, obtaining

$$\begin{aligned}&\frac{1}{N}\log\frac{Q_N(\nu, h, \alpha)}{Q_N(\nu, h, 0)} \\ &\quad = \frac{1}{N}\log\langle e^{\log\phi(x)}\rangle \\ &\quad = \alpha - \alpha^2\frac{\langle x_1^4\rangle(h, N)}{2} + \alpha^3\frac{\langle x_1^6\rangle(h, N)}{3} \\ &\qquad - \alpha^4\left[\frac{\langle x_1^8\rangle(h, N)}{4} - \frac{\langle V_4^2\rangle(h, N)}{8N}\right] + \alpha^5\left[\frac{\langle x_1^{10}\rangle(h, N)}{5} - \frac{\langle V_4V_6\rangle(h, N)}{6N}\right] \\ &\qquad - \alpha^6\left[\frac{\langle x_1^{12}\rangle(h, N)}{6} - \frac{\langle V_4V_8\rangle(h, N)}{8N} - \frac{\langle V_6^2\rangle(h, N)}{18N} + \frac{\langle V_4^3\rangle(h, N)}{48N}\right] \\ &\qquad + \cdots\end{aligned} \tag{4}$$

where

$$V_m = \sum_1^N x_k^m - N\langle x_1^m\rangle(h, N)$$

and we have written $\langle\ \rangle(h, N)$ to emphasize the dependence of the spherical averages on h and N.

We now assume ρ is a general one-dimensional interaction. Then

$$\rho_{ij} = \rho(|\mathbf{r}_i - \mathbf{r}_j|) = \rho(|i - j|) \tag{5}$$

We are assuming that the one-dimensional lattice we are now considering is periodic. Thus $\rho(N-1) = \rho(1)$ since the first and Nth sites are next to each other, and in general

$$\rho(k) = \rho(N-k), \qquad k = 1, 2, \ldots, N-1 \tag{6}$$

Finally, assume that

$$\sum_{j=-\infty}^{\infty} \rho(j) < \infty$$

and let

$$g(\theta) = \sum_{j=-\infty}^{\infty} \rho(j) e^{ij\theta}$$

Let

$$\nu_c = \frac{1}{2\pi}\int_0^{2\pi} \frac{d\theta}{g(0) - g(\theta)} \tag{7}$$

There will be a phase transition for $\nu_c < \infty$. It now follows from Eq. (4), Appendix A, and the long and tedious calculations of Appendix B, that[3]

$$\lim_{h\to 0}\lim_{N\to 0} \frac{1}{N}\log\frac{Q_N(\nu, h, \alpha)}{Q_N(\nu, h, 0)}$$

$$= \begin{cases} \alpha - \frac{3}{2}\alpha^2 + 5\alpha^3 - \left[\frac{105}{4} - 3\sum_{n=-\infty}^{\infty}\left(\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{in\theta}}{2s^* - \nu g(\theta)}\right)^4\right]\alpha^4 + \cdots \\ \qquad\qquad \text{for} \quad \nu \leqslant \nu_c \\ \alpha - \frac{1}{2}\left[3 - 2\left(1 - \frac{\nu_c}{\nu}\right)^2\right]\alpha^2 + \frac{1}{3}\left[15 - 30\left(1 - \frac{\nu_c}{\nu}\right)^2 + 16\left(1 - \frac{\nu_c}{\nu}\right)^3\right]\alpha^3 \\ \quad - \left[\frac{105}{4} - 105\left(1 - \frac{\nu_c}{\nu}\right)^2 + 112\left(1 - \frac{\nu_c}{\nu}\right)^3 - 33\left(1 - \frac{\nu_c}{\nu}\right)^4\right. \\ \quad - \frac{3}{\nu^4}\sum_{n=-\infty}^{\infty}\left(\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{in\theta}\,d\theta}{g(0) - g(\theta)}\right)^4 \\ \quad - \frac{12}{\nu^3}\left(1 - \frac{\nu_c}{\nu}\right)\sum_{n=-\infty}^{\infty}\left(\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{in\theta}\,d\theta}{g(0) - g(\theta)}\right)^3 \\ \quad \left. - \frac{4}{\nu^2}\left(1 - \frac{\nu_c}{\nu}\right)^2\sum_{n=-\infty}^{\infty}\left(\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{in\theta}\,d\theta}{g(0) - g(\theta)}\right)^2\right]\alpha^4 + \cdots \\ \qquad\qquad \text{for} \quad \nu \geqslant \nu_c \end{cases} \tag{8}$$

[3] s^* is determined by $(1/2\pi)\int_0^{2\pi}\{d\theta/[2s^* - \nu g(\theta)]\} = 1$.

If the calculations are performed only for $\nu \leqslant \nu_c$ the limits can be found much simpler by first setting $h = 0$, but the calculations still become long and tedious at the sixth term. From Ref. 3 the fifth and sixth terms for $\nu \leqslant \nu_c$ are

$$\left[189 - 60 \sum_{-\infty}^{\infty} \left(\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{in\theta}\, d\theta}{2s^* - \nu g(\theta)}\right)^4\right]\alpha^5$$

$$-\left[\frac{3465}{2} - 930 \sum_{-\infty}^{\infty} \left(\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{in\theta}\, d\theta}{2s^* - \nu g(\theta)}\right)^4\right.$$

$$- 40 \sum_{-\infty}^{\infty} \left(\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{in\theta}\, d\theta}{2s^* - \nu g(\theta)}\right)^6 + 36 \sum_{n_1, n_2 = -\infty}^{\infty} \left(\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{in_1\theta}\, d\theta}{2s^* - \nu g(\theta)}\right)^2$$

$$\left.\times \left(\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{in_2\theta}\, d\theta}{2s^* - \nu g(\theta)}\right)^2 \left(\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{-i(n_1+n_2)\theta}\, d\theta}{2s^* - \nu g(\theta)}\right)^2\right]\alpha^6$$

From (8) we see that the terms in the series are analytic functions of ν except at $\nu = \nu_c$, provided, of course, that all the sums converge. This ν_c, defined by (7), and the saddle point equation

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{d\theta}{2s^* - \nu g(\theta)} = 1$$

are the same as for the spherical model. In particular, ν_c *does not depend* on α, as mentioned at the end of the introduction.

If the above limit were calculated in d space dimensions instead of one, then, as in the spherical model, each $1/2\pi$ should be replaced by $1/(2\pi)^d$, each $\int_0^{2\pi}$ by d integrals $\int_0^{2\pi}$, and $g(\theta)$ should be replaced by the multiple Fourier series $g(\boldsymbol{\theta}) = \sum_{\mathbf{j}} \rho(\mathbf{j}) \exp(i\boldsymbol{\theta}\cdot\mathbf{j})$, where the sum extends over the infinite d-dimensional lattice, and so on.

It is now natural to ask whether the series given by (8) converges for sufficiently small α. In general the answer is no. In fact, the coefficient of α^4 in (8) for $\nu \geqslant \nu_c$ may very well diverge. For example, if

$$g(0) - g(\theta) = \sum_{n=1}^{\infty} \frac{1}{n^{1+r}} (1 - \cos n\theta)$$

then $g(0) - g(\theta) \sim \theta^r$. Hence $\nu_c < \infty$ if $r < 1$ but

$$\sum_{n=-\infty}^{\infty} \left(\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{in\theta}\, d\theta}{g(0) - g(\theta)}\right)^2 = \frac{1}{2\pi}\int_0^{2\pi} \frac{d\theta}{[g(0) - g(\theta)]^2}$$

diverges for $r \geqslant \frac{1}{2}$. The other two coefficients of α^4 for $\nu \geqslant \nu_c$ diverge for $r \geqslant \frac{3}{4}$ and $r \geqslant \frac{2}{3}$. Even when there is no phase transition ($\nu_c = \infty$) and the limit is given by the first part of (8) one might wonder about the convergence of the series, for although each term is finite, the coefficients of α^n increase with n. Here it is illuminating to take the simplest possible example, $\rho_{ij} \equiv 0$. Then the limit can be calculated explicitly and it turns out that the radius of convergence is $5 - \sqrt{24} \sim 0.1$, which is by no means large. In Ref. 3 we established the convergence of the series for some simple one-dimensional models with no phase transition, but even these proofs were not easy. We shall deal with these convergence questions in subsequent publications.

APPENDIX A. THE MATRIX FOR A GENERAL ONE-DIMENSIONAL INTERACTION

For a general one-dimensional interaction ρ we had [(5) and (6)]

$$\begin{aligned} \rho_{ij} &= \rho(|i-j|) \qquad \text{for all} \quad i, j \\ \rho(k) &= \rho(N-k), \qquad k = 1, 2, \ldots, N-1 \end{aligned} \tag{A.1}$$

Because ρ satisfies these conditions, it falls into a special class of matrices called cyclic matrices[1]. As a result all the eigenvalues and eigenvectors of (ρ_{ij}) may be written down explicitly.

The matrix of eigenvectors is

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ r_0 & r_1 & \cdots & r_{N-1} \\ r_0{}^2 & r_1{}^2 & \cdots & r_{N-1}^2 \\ \vdots & & & \vdots \\ r_0^{N-1} & r_1^{N-1} & \cdots & r_{N-1}^{N-1} \end{pmatrix}$$

where $r_k = e^{2\pi ik/N}$ is a root of unity, and the eigenvalues are

$$\lambda_k = \sum_{j=0}^{N-1} \rho(j) e^{2\pi ijk/N}, \qquad k = 0, 1, \ldots, N-1$$

We set $\rho(-j) = \rho(j)$ and rewrite λ_k as

$$\lambda_k = \sum_{j=-(N-1)/2}^{(N-1)/2} \rho(j) e^{2\pi ijk/N}, \qquad k = 0, 1, \ldots, N-1, \quad N \text{ odd} \tag{A.2}$$

with a similar formula for N even. By writing λ_k in this last form we have removed the implicit dependence of $\rho(j)$ on N, which came from the equation $\rho(j) = \rho(N-j)$.

We also note that $|\lambda_k| \leqslant \lambda_0$ for $k = 0, 1, \ldots, N-1$, so λ_0 is the maximum eigenvalue of $\rho(i-j)$.

The matrix $A = (2sI - \nu\rho(i-j))^{-1}$ will be important in the calculations in Appendix B. Since $\rho(i-j)$ is cyclic, so is the matrix $(2sI - \nu\rho(i-j))$ and hence A also, because the inverse of a cyclic matrix is cyclic. Thus the elements a_{ij} of A can be written

$$a_{ij} = p(|i-j|) = p_{i-j} \tag{A.3}$$

where $p(k) = p(N-k)$ for $k = 1, 2, \ldots, N-1$. Then $p_n = p_{i-j} = a_{ij}$ can be calculated from the eigenvalues and normalized eigenvectors of $(2sI - \nu\rho(i-j))$. The result is

$$p_n = \frac{1}{N}\sum_{k=0}^{N-1}\frac{e^{2\pi ink/N}}{2s - \nu\lambda_k}, \qquad i = \sqrt{-1} \tag{A.4}$$

APPENDIX B. THE CALCULATION OF THE SPHERICAL MODEL CORRELATION FUNCTIONS

1. We begin by reviewing the calculation of the spherical model partition function in a nonzero magnetic field, which is given by

$$Q_N(\nu, h) = \int_{\sum_1^N x_k^2 = N} \exp\left[\tfrac{1}{2}\nu \sum_{i,j=1}^{N} \rho(i-j)x_i x_j + h\sum_1^N x_i\right] d\sigma_{\sqrt{N}} \tag{B.1}$$

We introduce the corresponding integral over all space

$$\tilde{Q}_N(s, \nu, h) = \int_{-\infty}^{\infty} \cdots \int \exp\left[-s\sum_1^N x_k^2 + \tfrac{1}{2}\nu\sum_{i,j=1}^{N}\rho(i-j)x_i x_j + h\sum_1^N x_i\right] dx_1 \cdots dx_N$$

$$\operatorname{Re} s > \tfrac{1}{2}\nu\lambda_0 \tag{B.2}$$

This can be written

$$\tilde{Q}_N(s, \nu, h) = \int_0^{\infty} [\exp(-sr^2)]\left\{\int_{\sum_1^N x_k^2 = r^2} \exp\left[\tfrac{1}{2}\nu\sum \rho(i-j)x_i x_j + h\sum_1^N x_i\right] d\sigma_r\right\} dr$$

Changing variables by letting $t = r^2$, we have

$$\tilde{Q}_N(s, \nu, h) = \int_0^{\infty} [\exp(-st)]\left\{\frac{1}{2\sqrt{t}}\int_{\sum_1^N x_k^2 = t} \exp\left[\frac{\nu}{2}\sum \rho(i-j)x_i x_j + h\sum_l^N x_i\right] d\sigma_{\sqrt{t}}\right\} dt$$

This is just a Laplace transform. By the Laplace inversion formula,

$$\frac{1}{2\sqrt{t}}\int_{\sum_1^N x_k^2 = t} \exp\left[\frac{\nu}{2}\sum \rho(i-j)x_i x_j + h\sum_1^N x_i\right] d\sigma_{\sqrt{t}} = \frac{1}{2\pi i}\int_{s_0 - i\infty}^{s_0 + i\infty} e^{st}\tilde{Q}_N(s, \nu, h)\, ds$$

We now let $t = N$. Then

$$Q_N(\nu, h) = \frac{\sqrt{N}}{\pi i}\int_{s_0-i\infty}^{s_0+i\infty} e^{Ns}\tilde{Q}_N(s, \nu, h)\, ds \tag{B.3}$$

Evaluating the Gaussian integral in (B.2), we obtain

$$\tilde{Q}_N(s, \nu, h) = \left[\frac{(2\pi)^N}{(2s - \nu\lambda_0)\cdots(2s - \nu\lambda_{N-1})}\right]^{1/2} \exp\left[\frac{1}{2}(A\mathbf{h}, \mathbf{h})\right]$$

where $\lambda_0,\ldots, \lambda_{N-1}$ are the eigenvalues of $(\rho(i - j))$ given by Eq. (A.2), A^{-1} is the matrix $(2sI - \nu\rho(i - j))$, and $\mathbf{h}$ is the N-vector $(h,\ldots, h)$. Since $\mathbf{1} = (1,\ldots, 1)$ is the eigenvector of A^{-1} with eigenvalue $2s - \nu\lambda_0$, it is an eigenvector of A with eigenvalue $1/(2s - \nu\lambda_0)$. Therefore, the quadratic form

$$(A\mathbf{h}, \mathbf{h}) = h^2 \frac{1}{2s - \nu\lambda_0}(\mathbf{1}, \mathbf{1}) = h^2 \frac{N}{2s - \nu\lambda_0}$$

Thus

$$\tilde{Q}_N(s, \nu, h) = \left[\frac{(2\pi)^N}{(2s - \nu\lambda_0)\cdots(2s - \nu\lambda_{N-1})}\right]^{1/2} \exp\left[\frac{Nh^2}{2(2s - \nu\lambda_0)}\right] \tag{B.4}$$

Substituting in (B.3), we obtain

$$Q_N(\nu, h) = \frac{[N(2\pi)^N]^{1/2}}{\pi i}\int_{s_0-i\infty}^{s_0+i\infty} \exp\left\{N\left[s - \frac{1}{2N}\sum_{k=0}^{N-1}\log(2s - \nu\lambda_k) + \frac{h^2}{2(2s - \nu\lambda_0)}\right]\right\} ds$$

The term in the exponent has a saddle point s_N given by

$$1 - \frac{1}{N}\sum_{k=0}^{N-1}\frac{1}{2s_N - \nu\lambda_k} - \frac{h^2}{(2s_N - \nu\lambda_0)^2} = 0 \tag{B.5}$$

Choosing the path of integration to go through the saddle point, noting that $(1/N)\sum_{k=0}^{N-1}\log(2s - \nu\lambda_k)$ and $(1/N)\sum_{k=0}^{N-1}[1/(2s_N - \nu\lambda_k)]$ are Riemann sums, and letting $s^* = \lim_{N\to\infty} s_N$, we can evaluate the integral by the saddle point method, with the result

$$\begin{aligned} q(\nu, h) &= \lim_{N\to\infty}\frac{1}{N}\log Q_N(\nu, h) \\ &= \frac{1}{2}\log 2\pi + s^* - \frac{1}{2(2\pi)}\int_0^{2\pi}\log[2s^* - \nu g(\theta)]\, d\theta + \frac{h^2}{2[2s^* - \nu g(0)]} \end{aligned}$$

where s^* is determined by

$$1 = \frac{1}{2\pi}\int_0^{2\pi}\frac{d\theta}{2s^* - \nu g(\theta)} + \frac{h^2}{[2s^* - \nu g(0)]^2} \tag{B.6}$$

and

$$g(\theta) = \sum_{j=-\infty}^{\infty}\rho(j)e^{ij\theta} \tag{B.7}$$

There will be a phase transition for the interaction $\rho(i-j)$ if $\int_0^{2\pi}\{d\theta/[g(0)-g(\theta)]\}<\infty$. The saddle point equation (B.6) determines s^* as a function of h, $s^*(h)$. Note that $s^*(h)$ is an increasing function of h and $s^*(h)>\frac{1}{2}\nu g(0)$ for all $h>0$. Therefore $\lim_{h\to 0}s^*(h)$ exists. Let

$$\nu_c=\frac{1}{2\pi}\int_0^{2\pi}\frac{d\theta}{g(0)-g(\theta)} \tag{B.8}$$

and consider two cases:

I. $\nu<\nu_c$. Then from (B.6) we see that $\lim_{h\to 0}s^*(h)\neq\frac{1}{2}\nu g(0)$.
II. $\nu\geqslant\nu_c$. Then from (B.6) we see that $\lim_{h\to 0}s^*(h)=\frac{1}{2}\nu g(0)$.

Therefore from (B.6) and the above

$$\lim_{h\to 0}\frac{h^2}{[2s^*-\nu g(0)]^2}=\begin{cases}0, & \nu<\nu_c\\ 1-(\nu_c/\nu), & \nu\geqslant\nu_c\end{cases} \tag{B.9}$$

2. We now show how to calculate the spherical model correlation functions. If $F(x_1,\ldots,x_N)$ is any function of $x_1,\ldots,x_N$, we define

$$\begin{aligned}&\langle F(x_1,\ldots,x_N)\rangle(h,N)\\&=\left\{\int_{\sum_1^N x_k^2=N}F(x_1,\ldots,x_N)\exp\left[\frac{\nu}{2}\sum_{i,j=1}^N\rho(i-j)x_ix_j+h\sum_l^N x_i\right]d\sigma_{\sqrt{N}}\right\}\\&\quad\times\left\{\int_{\sum_1^N x_k^2=N}\exp\left[\frac{\nu}{2}\sum_{i,j=1}^N\rho(i-j)x_ix_j+h\sum_l^N x_i\right]d\sigma_{\sqrt{N}}\right\}^{-1}\end{aligned} \tag{B.10}$$

and

$$\begin{aligned}&\langle F(x_1,\widetilde{\ldots},x_N)\rangle(h,N)\\&=\left\{\int_{-\infty}^{\infty}\cdots\int F(x_1,\ldots,x_N)\right.\\&\quad\left.\times\exp\left[-s\sum_1^N x_k^2+\frac{\nu}{2}\sum\rho(i-j)x_ix_j+h\sum_1^N x_i\right]dx_1\cdots dx_N\right\}\\&\quad\times\left\{\int_{-\infty}^{\infty}\cdots\int\exp\left[-s\sum_1^N x_k^2+\frac{\nu}{2}\sum\rho(i-j)x_ix_j\right.\right.\\&\quad\left.\left.+h\sum_1^N x_i\right]dx_1\cdots dx_N\right\}^{-1}\end{aligned} \tag{B.11}$$

Then, similar to (B.3), we have

$$\begin{aligned}&\langle F(x_1,\ldots,x_N)\rangle(h,N)\\&=\left[\int_{s_0-i\infty}^{s_0+i\infty}e^{Ns}\tilde{Q}_N\langle F(x_1,\widetilde{\ldots},x_N)\rangle(h,N)\,ds\right]\left(\int_{s_0-i\infty}^{s_0+i\infty}e^{Ns}\tilde{Q}_N\,ds\right)^{-1}\end{aligned} \tag{B.12}$$

If we write (B.11) as

$$\langle F(x_1,\ldots, x_N)\rangle(h, N)$$
$$= \left\{\int_{-\infty}^{\infty} \cdots \int F(x_1,\ldots, x_N) \exp[(\mathbf{h}, \mathbf{x})] \exp[-\tfrac{1}{2}(A^{-1}\mathbf{x}, \mathbf{x})]\, dx_1 \cdots dx_N\right\}$$
$$\times \left\{\int_{-\infty}^{\infty} \cdots \int \exp[(\mathbf{h}, \mathbf{x})] \exp[-\tfrac{1}{2}(A^{-1}\mathbf{x}, \mathbf{x})]\, dx_1 \cdots dx_N\right\}^{-1} \tag{B.12$'$}$$

and make the change of variables

$$y_i = x_i - t$$

where $\mathbf{t} = (t,\ldots, t)$ is given by

$$\mathbf{t} = A\mathbf{h} \tag{B.13}$$

we see that

$$\langle F(x_1,\ldots, x_N)\rangle(h, N) = \langle F(y_1 + t,\ldots, y_N + t)\rangle(0, N)$$

Writing $\langle\ \rangle(N)$ for $\langle\ \rangle(0, N)$ and replacing the y's by x's, we obtain

$$\langle F(x_1,\ldots, x_N)\rangle(h, N) = \langle F(x_1 + t,\ldots, x_N + t)\rangle(N) \tag{B.14}$$

3. Now consider $F(x_1,\ldots, x_N) = x_1{}^m$, where m is a positive, even integer. The method for calculating the correlation function for a single variable is already known, (1) but is included here for the sake of clarity. Then

$$\langle x_1{}^m\rangle(h, N) = \langle(x_1 + t)^m\rangle(N) = \sum_{k=0}^{m} \binom{m}{k} t^k \langle x_1^{m-k}\rangle(N)$$

Since

$$\langle x_1^{m-k}\rangle(N) = \begin{cases} 0, & k \text{ odd} \\ (m-k-1)!!\, \langle x_1{}^2\rangle^{(m-k)/2}(N), & k \text{ even} \end{cases}$$

where $(m - k - 1)!! = (m - k - 1)(m - k - 3) \cdots 5\cdot 3\cdot 1$, then

$$\langle x_1{}^m\rangle(h, N) = \sum_{\substack{0\le k\le m \\ k \text{ even}}} \binom{m}{k} (m-k-1)!!\, t^k \langle x_1{}^2\rangle^{(m-k)/2}(N) \tag{B.15}$$

From (B.13)

$$t = h \sum_{j=1}^{N} a_{ij}, \qquad \text{for} \quad i = 1,\ldots, N, \qquad \text{where} \quad A = (a_{ij})$$

or

$$t = \frac{1}{N} h \sum_{i,j=1}^{N} a_{ij} = \frac{1}{N} h(A\mathbf{1}, \mathbf{1}) = \frac{h}{2s - \nu\lambda_0} \tag{B.16}$$

and

$$\lim_{N\to\infty} t = \frac{h}{[2s - \nu g(0)]} \tag{B.17}$$

Since $\langle\widetilde{x_1^2}\rangle(N) = \langle\widetilde{x_i^2}\rangle(N)$ for $i = 1,\ldots, N$ because A is cyclic,

$$\langle\widetilde{x_1^2}\rangle(N) = \frac{1}{N}\left\langle\sum_1^N \widetilde{x_i^2}\right\rangle(N) = -\frac{1}{N}\frac{\partial}{\partial s}\log \tilde{Q}_N(s, \nu, 0)$$

From Eq. (B.4) this is

$$\langle\widetilde{x_1^2}\rangle(N) = \frac{1}{N}\sum_{k=0}^{N-1}\frac{1}{2s - \nu\lambda_k} \tag{B.18}$$

Substituting $F(x_1,\ldots, x_m) = x_1^m$ in (B.12) and using (B.15) gives the formula for the spherical correlation function

$$\langle x_1^m\rangle(h, N) = \sum_{\substack{0\le k\le m\\ k\text{ even}}}\binom{m}{k}(m-k-1)!!\,\frac{\int_{s_0-i\infty}^{s_0+i\infty} e^{Ns}\tilde{Q}_N t^k \langle\widetilde{x_1^2}\rangle^{(m-k)/2}(N)\, ds}{\int_{s_0-i\infty}^{s_0+i\infty} e^{Ns}\tilde{Q}_N\, ds}$$

As before the exponent in $\exp\{N[s + (1/N)\log \tilde{Q}_N]\}$ has the saddle point s_N given by (B.5). Therefore

$$\langle x_1^m\rangle(h, N) \sim \sum_{\substack{0\le k\le m\\ k\text{ even}}}\binom{m}{k}(m-k-1)!!\left(\frac{h}{2s_N - \nu\lambda_0}\right)^k\left(\frac{1}{N}\sum_{k=0}^{N-1}\frac{1}{2s_N - \nu\lambda_k}\right)^{(m-k)/2}$$

where we have substituted for t and $\langle\widetilde{x_1^2}\rangle(N)$ from (B.16) and (B.18). Then

$$\mu_m(\nu, h) = \lim_{N\to\infty}\langle x_1^m\rangle(h, N) = \sum_{\substack{0\le k\le m\\ k\text{ even}}}\binom{m}{k}(m-k-1)!!\left(\frac{h}{2s^* - \nu g(0)}\right)^k\left(\frac{1}{2\pi}\int_0^{2\pi}\frac{d\theta}{2s^* - \nu g(\theta)}\right)^{(m-k)/2} \tag{B.19}$$

where s^* is determined by (B.6). It follows from (B.6) and (B.9) that

$$\lim_{h\to 0}\frac{1}{2\pi}\int_0^{2\pi}\frac{d\theta}{2s^* - \nu g(\theta)} = \begin{cases}1, & \nu < \nu_c\\ \nu_c/\nu, & \nu \geqslant \nu_c\end{cases} \tag{B.20}$$

Finally passing to the limit as $h \to 0$ in (B.19) and using (B.9) and (B.20), we obtain

$$\mu_m(\nu) = \lim_{h\to 0}\mu_m(\nu, h) = \begin{cases}\displaystyle\sum_{\substack{0\le k\le m\\ k\text{ even}}}\binom{m}{k}(m-k-1)!!\left(1 - \frac{\nu_c}{\nu}\right)^{k/2}\left(\frac{\nu_c}{\nu}\right)^{(m-k)/2}, & \nu \geqslant \nu_c\\ (m-1)!!, & \nu \leqslant \nu_c\end{cases} \tag{B.21}$$

For example,

$$\mu_2(\nu) = 1\frac{\nu_c}{\nu} + 1\left(1 - \frac{\nu_c}{\nu}\right) = 1, \qquad \nu \geqslant \nu_c$$

$$\mu_4(\nu) = 3\left(\frac{\nu_c}{\nu}\right)^2 + 6\frac{\nu_c}{\nu}\left(1 - \frac{\nu_c}{\nu}\right) + \left(1 - \frac{\nu_c}{\nu}\right)^2 = 3 - 2\left(1 - \frac{\nu_c}{\nu}\right)^2, \qquad \nu \geqslant \nu_c$$

The formulas for the first four nonzero $\mu_m(\nu)$, obtained from (B.21), are

$$\mu_2(\nu) = 1 \qquad \text{all } \nu$$

$$\mu_4(\nu) = \begin{cases} 3, & \nu \leqslant \nu_c \\ 3 - 2\left(1 - \frac{\nu_c}{\nu}\right)^2, & \nu \geqslant \nu_c \end{cases}$$

$$\mu_6(\nu) = \begin{cases} 15, & \nu \leqslant \nu_c \\ 15 - 30\left(1 - \frac{\nu_c}{\nu}\right)^2 + 16\left(1 - \frac{\nu_c}{\nu}\right)^3, & \nu \geqslant \nu_c \end{cases} \tag{B.22}$$

$$\mu_8(\nu) = \begin{cases} 105, & \nu \leqslant \nu_c \\ 105 - 420\left(1 - \frac{\nu_c}{\nu}\right)^2 + 448\left(1 - \frac{\nu_c}{\nu}\right)^3 - 132\left(1 - \frac{\nu_c}{\nu}\right)^4, & \nu \geqslant \nu_c \end{cases}$$

Except for $\mu_2(\nu)$, each $\mu_m(\nu)$ shows a break in analyticity at the phase transition point ν_c. We have $\mu_2 \equiv 1$ necessarily because the spherical moments are computed integrating over the sphere $\sum_1^N x_k^2 = N$. Also, for each m, $\mu_m(\nu) \to 1$ as $\nu \to \infty$, corresponding to perfect order in the magnet at zero temperature.

4. Pair correlations. Finally we consider $F(x_1, \ldots, x_N) = x_i^m x_j^n$, where m and n are positive, even integers. Then, similar to the above,

$$\begin{aligned} \langle \widetilde{x_i^m x_j^n} \rangle(h, N) &= \langle \widetilde{(x_i + t)^m (x_j + t)^n} \rangle(N) \\ &= \sum_{k=0}^{m} \sum_{l=0}^{n} \binom{m}{k} \binom{n}{l} \langle \widetilde{x_i^{m-k} x_j^{n-l}} \rangle(N) t^{k+l} \end{aligned} \tag{B.23}$$

In order to compute the limit as $N \to \infty$ and $h \to 0$ of $\langle V_4^2\rangle(h, N)/8N$ in Eq. (4) we need to calculate $\langle \widetilde{x_i^4 x_j^4}\rangle(h, N)$, i.e., the case $m = n = 4$. Although this is the simplest case, it is highly nontrivial and the remainder of the paper is devoted to its calculation. Since $k + l$ must be even for $\langle \widetilde{x_i^{4-k} x_j^{4-l}} \rangle(N)$ to be nonzero, the terms that contribute to $\langle \widetilde{x_i^4 x_j^4}\rangle(h, N)$ are

$$\langle \widetilde{x_i x_j} \rangle(N), \quad \langle \widetilde{x_i^2 x_j^2} \rangle(N), \quad \langle \widetilde{x_i^3 x_j^3} \rangle(N), \quad \langle \widetilde{x_i^4 x_j^4} \rangle(N)$$

$$\langle \widetilde{x_i^3 x_j} \rangle(N), \quad \langle \widetilde{x_i^4 x_j^2} \rangle(N), \quad \langle \widetilde{x_i x_j^3} \rangle(N), \quad \langle \widetilde{x_i^2 x_j^4} \rangle(N)$$

plus

$$\widetilde{\langle x_i^2\rangle}(N),\quad \widetilde{\langle x_i^4\rangle}(N),\quad \widetilde{\langle x_j^2\rangle}(N),\quad \widetilde{\langle x_j^4\rangle}(N)$$

which we have already computed.

From (B.12′), $a_{ij} = \widetilde{\langle x_i x_j\rangle}(0, N) = \widetilde{\langle x_i x_j\rangle}(N)$ and using the formula

$$\langle x_1^{\gamma_1} \tilde{\cdots}\, x_N^{\gamma_N}\rangle(N) = \langle x_1 \tilde{\cdots}\, x_N\rangle \quad \text{written individually}$$
$$= \sum_{\text{pairings}} \prod_{\text{pairs}} \widetilde{\langle x_i x_j\rangle}$$

for Gaussian distributions with mean zero, we find

$$\begin{aligned}
&\widetilde{\langle x_i^2 x_j^2\rangle} = a_{ii}a_{jj} + 2a_{ij}^2, \qquad \widetilde{\langle x_i^3 x_j^3\rangle} = 9a_{ii}a_{ij}a_{jj} + 6a_{ij}^3\\
&\widetilde{\langle x_i^4 x_j^4\rangle} = 9a_{ii}^2a_{jj}^2 + 72a_{ii}a_{ij}^2a_{jj} + 24a_{ij}^4\\
&\widetilde{\langle x_i^3 x_j\rangle} = 3a_{ii}a_{ij}, \qquad \widetilde{\langle x_i^4 x_j^2\rangle} = 3a_{ii}^2a_{jj} + 12a_{ij}^2, \qquad \text{etc.}
\end{aligned} \tag{B.24}$$

It follows from (A.3) that we can write $a_{ij} = p(|i - j|) = p_{i-j}$, where $p(k) = p(N - k)$ for $k = 1, 2,..., N - 1$.

Then the above formulas simplify to

$$\begin{aligned}
&\widetilde{\langle x_i x_j\rangle} = p_{i-j}, && \widetilde{\langle x_i^2 x_j^2\rangle} = p_0^2 + 2p_{i-j}^2\\
&\widetilde{\langle x_i^3 x_j^3\rangle} = 9p_0^2p_{i-j} + 6p_{i-j}^3, && \widetilde{\langle x_i^4 x_j^4\rangle} = 9p_0^4 + 72p_0^2p_{i-j}^2 + 24p_{i-j}^4\\
&\widetilde{\langle x_i^3 x_j\rangle} = \widetilde{\langle x_i x_j^3\rangle} = 3p_0p_{i-j}, && \widetilde{\langle x_i^4 x_j^2\rangle} = \widetilde{\langle x_i^2 x_j^4\rangle} = 3p_0^3 + 12p_0p_{i-j}^2
\end{aligned} \tag{B.25}$$

and also $\widetilde{\langle x_i^m\rangle} = (m - 1)!!\, p_0^{m/2}$. Now if we let the bar denote the inverse of the tilde and let $\mu = t^2$, we have

$$\begin{aligned}
\langle V_4^2\rangle(h, N) &= \sum_{i,j=1}^{N} \langle x_i^4 x_j^4\rangle(h, N) - N^2\langle x_1^4\rangle^2(h, N)\\
&= \sum_{i,j=1}^{N} \overline{\widetilde{\langle x_i^4 x_j^4\rangle}}(h, N) - N^2(\overline{\widetilde{\langle x_1^4\rangle}}(h, N))^2\\
&= \sum_{k=0}^{4}\sum_{l=0}^{4} \binom{4}{k}\binom{4}{l} \sum_{i,j=1}^{N} \overline{\widetilde{\langle x_i^{m-k} x_j^{n-l}\rangle}(N)\mu^{(k+l)/2}}\\
&\quad - N^2\left[\sum_{k=0,2,4} \binom{4}{k}(m - k - 1)!!\, \overline{\widetilde{\langle x_i^2\rangle}^{(4-k)/2}\mu^{k/2}}\right]^2
\end{aligned}$$

from (B.15) and (B.23). Expanding these sums, substituting from (B.25), and simplifying, we obtain

$$\begin{aligned}\langle V_4{}^2\rangle(h, N) = {} & 9N^2(\overline{p_0{}^4} - \overline{p_0^2}^2) + 36N^2(\overline{p_0{}^2\mu^2} - \overline{p_0\mu}^2) \\ & + N^2(\overline{\mu^4} - \overline{\mu^2}^2) + 6N^2(\overline{p_0{}^2\mu^2} - \overline{p_0{}^2}\,\overline{\mu^2}) \\ & + 36N^2(\overline{p_0{}^3\mu} - \overline{p_0{}^2}\,\overline{p_0\mu}) + 12N^2(\overline{p_0\mu^3} - \overline{p_0\mu}\,\overline{\mu^2}) \\ & + 72N\sum_n \overline{p_0{}^2p_n{}^2} + 24N\sum_n \overline{p_n{}^4} \\ & + 144N\sum_n \overline{p_0{}^2p_n\mu} + 96N\sum_n \overline{p_n{}^3\mu} \\ & + 72N\sum_n \overline{p_n{}^2\mu^2} + 16N\sum_n \overline{p_n\mu^3} \\ & + 144N\sum_n \overline{p_0p_n{}^2\mu} + 96N\sum_n \overline{p_0p_n\mu^2}\end{aligned} \tag{B.26}$$

where $\sum_n$ means $\sum_{n=-(N-1)/2}^{(N-1)/2}$ and N is odd.

From (A.4)

$$p_n = \frac{1}{N}\sum_{k=0}^{N-1}\frac{e^{2\pi i n k/N}}{2s - \nu\lambda_k} \tag{B.27}$$

Substituting for λ_k from (A.2), we get a Riemann sum. Then under suitable conditions on ρ,

$$\lim_{N\to\infty} p_n = \frac{1}{2\pi}\int_0^{2\pi}\frac{e^{in\theta}\,d\theta}{2s - \nu g(\theta)} \tag{B.28}$$

where $g(\theta)$ is given by (B.7). Since $\mu = t^2$, from (B.16) and (B.17)

$$\mu = \left(\frac{h}{2s - \nu\lambda_0}\right)^2 \tag{B.29}$$

and

$$\lim_{N\to\infty}\mu = \left(\frac{h}{2s - \nu g(0)}\right)^2 \tag{B.30}$$

Since the bar is the inverse of the tilde, it follows from (B.12) that

$$\overline{p_0{}^l p_n{}^r \mu^m} = \frac{\displaystyle\int_{s_0-i\infty}^{s_0+i\infty} e^{Ns}\tilde{Q}_N p_0{}^l p_n{}^r \mu^m\,ds}{\displaystyle\int_{s_0-i\infty}^{s_0+i\infty} e^{Ns}\tilde{Q}_N\,ds} \tag{B.31}$$

where l, r, and m are nonnegative integers. If we let

$$f(s) = s + (1/N)\log\tilde{Q}_N \tag{B.32}$$

and set $f'(s) = 0$, we have as before the saddle point s_N determined from Eq. (B.5). By the saddle point method

$$\overline{p_0{}^l p_n{}^r \mu^m} \sim p_0{}^l p_n{}^r \mu^m|_{s=s_N}$$
$$= \left(\frac{1}{N}\sum_{k=0}^{N-1}\frac{1}{2s_N - \nu\lambda_k}\right)^l \left(\frac{1}{N}\sum_{k=0}^{N-1}\frac{e^{2\pi ink/N}}{2s_N - \nu\lambda_k}\right)^r \left(\frac{h}{2s - \nu\lambda_0}\right)^{2m} \qquad \text{(B.33)}$$

from (B.27) and (B.29).

Then, using (B.28) and (B.30), we have, similarly to (B.19),

$$\lim_{N\to\infty} \overline{p_0{}^l p_n{}^r \mu^m}$$
$$= \left(\frac{1}{2\pi}\int_0^{2\pi}\frac{d\theta}{2s^* - \nu g(\theta)}\right)^l \left(\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{in\theta}\, d\theta}{2s^* - \nu g(\theta)}\right)^r \left(\frac{h}{2s^* - \nu g(0)}\right)^{2m} \qquad \text{(B.34)}$$

with s^* determined by (B.6).

Letting $h \to 0$ gives, similarly to (B.21) [see footnote preceding Eq. (8) for the condition determining s^*]

$$\lim_{h\to 0}\lim_{N\to\infty} \overline{p_0{}^l p_n{}^r \mu^m}$$
$$= \begin{cases} \left(\dfrac{1}{2\pi}\displaystyle\int_0^{2\pi}\dfrac{e^{in\theta}\, d\theta}{2s^* - \nu g(\theta)}\right)^r \delta_{m0}, & \nu \leqslant \nu_c \\ \dfrac{\nu_c{}^l}{\nu^{l+r}}\left(\dfrac{1}{2\pi}\displaystyle\int_0^{2\pi}\dfrac{e^{in\theta}\, d\theta}{g(0) - g(\theta)}\right)^r \left(1 - \dfrac{\nu_c}{\nu}\right)^m, & \nu \geqslant \nu_c \end{cases} \qquad \text{(B.35)}$$

Using this result and (B.26) gives [with the same condition for s^* as in (8) and (B.35)]

$$\lim_{h\to 0}\lim_{N\to\infty}\left(\text{7th} + \text{8th} + \text{10th} + \text{11th} + \text{13th} \quad \text{terms in } \frac{\langle V_4{}^2\rangle}{8N}\right)$$
$$= \begin{cases} 9\displaystyle\sum_{n=-\infty}^{\infty}\left(\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{in\theta}\, d\theta}{2s^* - \nu g(\theta)}\right)^2 + 3\sum_{n=-\infty}^{\infty}\left(\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{in\theta}\, d\theta}{2s^* - \nu g(\theta)}\right)^4, \\ \hfill \nu \leqslant \nu_c \\ 9\dfrac{\nu_c{}^2}{\nu^4}\displaystyle\sum_{n=-\infty}^{\infty}\left(\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{in\theta}\, d\theta}{g(0) - g(\theta)}\right)^2 + 3\frac{1}{\nu^4}\sum_{n=-\infty}^{\infty}\left(\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{in\theta}\, d\theta}{g(0) - g(\theta)}\right)^4 \\ + 12\dfrac{1}{\nu^3}\left(1 - \dfrac{\nu_c}{\nu}\right)\displaystyle\sum_{n=-\infty}^{\infty}\left(\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{in\theta}\, d\theta}{g(0) - g(\theta)}\right)^3 \\ + 9\dfrac{1}{\nu^2}\left(1 - \dfrac{\nu_c}{\nu}\right)^2\displaystyle\sum_{n=-\infty}^{\infty}\left(\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{in\theta}\, d\theta}{g(0) - g(\theta)}\right)^2 \\ + 18\dfrac{\nu_c}{\nu^3}\left(1 - \dfrac{\nu_c}{\nu}\right)\displaystyle\sum_{n=-\infty}^{\infty}\left(\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{in\theta}\, d\theta}{g(0) - g(\theta)}\right)^2, & \nu \geqslant \nu_c \end{cases} \qquad \text{(B.36)}$$

The second of the above expressions simplifies to

$$\frac{9}{\nu^2}\sum_{n=-\infty}^{\infty}\left(\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{in\theta}\,d\theta}{g(0)-g(\theta)}\right)^2 + \frac{12}{\nu^3}\left(1-\frac{\nu_c}{\nu}\right)\sum_{n=-\infty}^{\infty}\left(\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{in\theta}\,d\theta}{g(0)-g(\theta)}\right)^3 + \frac{3}{\nu^4}\sum_{n=-\infty}^{\infty}\left(\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{in\theta}}{g(0)-g(\theta)}\right)^4, \qquad \nu \geqslant \nu_c \tag{B.37}$$

The ninth, twelfth, and fourteenth terms must be treated separately. The ninth term in $\langle V_4^2\rangle/8N$ from (B.26) is

$$18\sum_n \overline{p_0{}^2 p_n \mu} = 18\overline{p_0{}^2\left(\sum_{n=-(N-1)/2}^{(N-1)/2} p_n\right)\mu}$$

But

$$\sum_{n=-(N-1)/2}^{(N-1)/2} p_n = \sum_{i-j=-(N-1)/2}^{(N-1)/2} p_{ij} = \sum_{j=1}^{N} a_{ij} \qquad \text{for} \quad i = 1,\ldots, N$$

$$= \frac{1}{N}\sum_{i,j=1}^{N} a_{ij} = \frac{1}{2s-\nu\lambda_0}$$

because of the cyclic nature of (a_{ij}) and by (B.16). Since $1/(2s-\nu\lambda_0) = \mu^{1/2}/h$, then $18\sum_n \overline{p_0{}^2 p_n\mu} = (18/h)\overline{p_0{}^2\mu^{3/2}}$. By (B.34)

$$\lim_{N\to\infty}\frac{18}{h}\overline{p_0{}^2\mu^{3/2}} = 18\left(\frac{1}{2\pi}\int_0^{2\pi}\frac{d\theta}{2s^*-\nu g(\theta)}\right)^2\frac{h^2}{[2s^*-\nu g(0)]^3} \tag{B.38}$$

and we see we cannot yet take the limit as $h \to 0$. The twelfth and fourteenth terms are handled the same way, and we find

$$\lim_{N\to\infty}\left(\text{9th} + \text{12th} + \text{14th} \quad \text{terms in } \frac{\langle V_4^2\rangle}{8N}\right)$$

$$= 18\left(\frac{1}{2\pi}\int_0^{2\pi}\frac{d\theta}{2s^*-\nu g(\theta)}\right)^2\frac{h^2}{[2s^*-\nu g(0)]^3} + 2\frac{h^6}{[2s^*-\nu g(0)]^7} + 12\left(\frac{1}{2\pi}\int_0^{2\pi}\frac{d\theta}{2s^*-\nu g(\theta)}\right)\frac{h^4}{[2s^*-\nu g(0)]^5} \tag{B.39}$$

Substituting

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{d\theta}{2s^*-\nu g(\theta)} = 1 - \frac{h^2}{[2s^*-\nu g(0)]^2}$$

from (B.6), this last expression becomes

$$18\frac{h^2}{[2s^*-\nu g(0)]^3} - 24\frac{h^4}{[2s^*-\nu g(0)]^5} + 8\frac{h^6}{[2s^*-\nu g(0)]^7} \tag{B.39'}$$

The above calculation can also be done by using (B.28) and noting

$$\lim_{N\to\infty} \sum_{n=-(N-1)/2}^{(N-1)/2} p_n = \sum_{n=-\infty}^{\infty} \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{in\theta}\, d\theta}{2s - \nu g(\theta)}$$
$$= \frac{1}{2\pi} \int_0^{2\pi} \frac{\delta(\theta)\, d\theta}{2s - \nu g(\theta)} = \frac{1}{2s - \nu g(0)}$$

We must now deal with the first six terms in (B.26). Since they are preceded by N^2 and we are only dividing by N, the terms of order 1 in $\overline{p_0{}^4} - \overline{p_0^2}^2$, etc., must vanish with terms of order $1/N$ contributing to the limit. This is indeed the case. To calculate the terms of order $1/N$, however, we must do the saddle point calculation to the next higher order.

In (B.31), setting $r = 0$, using (B.32), and changing the contour $(s_0 - i\infty, s_0 + i\infty)$ to Γ, the path of steepest descent through s_N, we obtain

$$\overline{p_0{}^l\mu^m} = \frac{\int_\Gamma e^{Nf(s)} p_0{}^l \mu^m\, ds}{\int_\Gamma e^{Nf(s)}\, ds} \tag{B.40}$$

Substituting from (B.4) into (B.32), we obtain

$$f(s) = \frac{1}{2}\log 2\pi + s - \frac{1}{2N}\sum_{k=0}^{N-1} \log(2s - \nu\lambda_k) + \frac{h^2}{2(2s - \nu\lambda_0)} \tag{B.41}$$

Along Γ, $\operatorname{Im} f(s) = 0$ and $\max_{s\in\Gamma} f(s) = f(s_N)$. Changing variables by letting

$$-w^2 = f(s) - f(s_N) \tag{B.42}$$

and substituting from (B.41), we obtain

$$-w^2 = s - s_N + \frac{1}{2}h^2 \frac{1}{2s_N - \nu\lambda_0}\left(\frac{1}{1 + [2(s - s_N)/(2s_N - \nu\lambda_0)} - 1\right)$$
$$- \frac{1}{2N}\sum_{k=0}^{N-1} \log\left(1 + \frac{2(s - s_N)}{2s_N - \nu\lambda_k}\right)$$

Expanding the right-hand side in a power series, we find

$$-w^2 = \left[\frac{2h^2}{(2s_N - \nu\lambda_0)^3} + \frac{1}{N}\sum_{k=0}^{N-1} \frac{1}{(2s_N - \nu\lambda_k)^2}\right](s - s_N)^2$$
$$- \left[\frac{4h^2}{(2s_N - \nu\lambda_0)^4} + \frac{4}{3}\frac{1}{N}\sum_{k=0}^{N-1} \frac{1}{(2s_N - \nu\lambda_k)^3}\right](s - s_N)^3 + \cdots \tag{B.43}$$

where the coefficient of $s - s_N$ vanishes by (B.5).

Let

$$\tau_{mn} = \frac{h^m}{(2s_N - \nu\lambda_0)^n} \tag{B.44}$$

and

$$\Lambda_m = \frac{1}{N}\sum_{k=0}^{N-1}\frac{1}{(2s_N - \nu\lambda_k)^m} \tag{B.45}$$

Then

$$w^2 = -(2\tau_{23} + \Lambda_2)(s - s_N)^2 + (4\tau_{24} + \tfrac{4}{3}\Lambda_3)(s - s_N)^3 - \cdots \tag{B.46}$$

and

$$w = -i(2\tau_{23} + \Lambda_2)^{1/2}(s - s_N)\left[1 - \frac{2\tau_{24} + \frac{2}{3}\Lambda_3}{2\tau_{23} + \Lambda_2}(s - s_N) + \cdots\right] \tag{B.47}$$

Let

$$\gamma = -i(2\tau_{23} + \Lambda_2)^{1/2} \tag{B.48}$$

and

$$D = (\tau_{24} + \tfrac{1}{3}\Lambda_3)/(2\tau_{23} + \Lambda_2) \tag{B.49}$$

Inverting (B.47), we obtain

$$s = s_N + \frac{w}{\gamma} + 2D\frac{w^2}{\gamma^2} + \cdots \tag{B.50}$$

$$\frac{ds}{dw} = \frac{1}{\gamma} + \frac{4D}{\gamma^2}w + \cdots \tag{B.51}$$

and

$$\frac{1}{ds/dw} = \gamma - 4Dw + \cdots \tag{B.52}$$

To complete the change of variables from s to w in (B.40) we need to solve for p_0 and μ in terms of w. Substituting (B.50) into $\mu = h^2/(2s - \nu\lambda_0)^2$, Eq. (B.29), and simplifying, we find

$$\mu = \mu_0\left[1 - \frac{4\mu_0}{h\gamma}w + \left(\frac{12\mu_0}{h^2\gamma^2} - \frac{8\mu_0^{1/2}}{h\gamma^2}D\right)w^2 + \cdots\right] \tag{B.53}$$

where

$$\mu_0 = \mu|_{s_N} = h^2/(2s_N - \nu\lambda_0)^2 \tag{B.53$'$}$$

From (B.42) and (B.32)

$$w^2 = -(s - s_N) + \frac{1}{N}\log \tilde{Q}_N|_{s_N} - \frac{1}{N}\log \tilde{Q}_N \tag{B.54}$$

Differentiating both sides of (B.54) with respect to w, we obtain

$$2w = -\frac{ds}{dw} - \frac{1}{N}\frac{\partial}{\partial s}\log \tilde{Q}_N \frac{ds}{dw} \tag{B.55}$$

From (B.4)

$$-\frac{1}{N}\frac{\partial}{\partial s}\log \tilde{Q}_N = \frac{1}{N}\sum_{k=0}^{N-1}\frac{1}{2s - \nu\lambda_k} + \frac{h^2}{(2s - \nu\lambda_0)^2} \tag{B.56}$$

By (B.27) and (B.29)

$$-\frac{1}{N}\frac{\partial}{\partial s}\log \tilde{Q}_N = p_0 + \mu$$

Substituting in (B.55) and solving for p_0, we obtain

$$p_0 = 1 - \mu + \frac{2w}{ds/dw} \tag{B.57}$$

Making the change of variables $w^2 = f(s_N) - f(s)$ in (B.40), we find

$$\overline{p_0{}^l\mu^m} = \frac{\int_{-\infty}^{\infty}[\exp(-Nw^2)]p_0{}^l\mu^m(ds/dw)\,dw}{\int_{-\infty}^{\infty}[\exp(-Nw^2)](ds/dw)\,dw} \tag{B.58}$$

because w goes from $-\infty$ to ∞, as s traverses the path Γ [Eq. (B.47)]. Substituting in (B.58) for p_0 using (B.57) and substituting for μ, $1/(ds/dw)$, and ds/dw from (B.53), (B.52), and (B.51), a tedious calculation and integration yields

$$\begin{aligned}\overline{p_0{}^l\mu^m} &= \mu_0{}^m(1 - \mu_0)^l + \frac{1}{N}\frac{\mu_0{}^m(1 - \mu_0)^{l-2}}{h\gamma^2}\\ &\times\Bigg\{12\mu_0^{1/2}(1 - \mu_0)D[l\mu_0 - m(1 - \mu_0)] + \frac{6[m(1 - \mu_0) - l\mu_0]\mu_0(1 - \mu_0)}{h}\\ &+ \frac{4[l(l - 1)\mu_0{}^2 + m(m - 1)(1 - \mu_0)^2]\mu_0}{h} - \frac{8ml\mu_0{}^2(1 - \mu_0)}{h}\\ &+ 4\gamma^2 l\mu_0^{1/2}[(l - 1)\mu_0 - m(1 - \mu_0)] + \gamma^4 hl(l - 1)\Bigg\} + \cdots\end{aligned} \tag{B.59}$$

We can now calculate the first six terms in (B.26) using this formula:

$$
\begin{aligned}
\overline{p_0^4} - \overline{p_0^2}^2 &= \frac{8}{N}\frac{(1-\mu_0)^2}{h\gamma^2}\left(\frac{4\mu_0^3}{h} + 4\gamma^2\mu_0^{3/2} + \gamma^4 h\right) + \cdots \\
\overline{p_0^2\mu^2} - \overline{p_0\mu}^2 &= \frac{2}{N}\frac{\mu_0^2}{h\gamma^2}\left[\frac{4\mu_0(2\mu_0-1)^2}{h} + 4\gamma^2\mu_0^{1/2}(2\mu_0-1) + \gamma^4 h\right] + \cdots \\
\overline{\mu^4} - \overline{\mu^2}^2 &= \frac{32\mu_0^5}{Nh^2\gamma^2} + \cdots \\
\overline{p_0^2\mu^2} - \overline{p_0^2}\,\overline{\mu^2} &= -\frac{16}{N}\frac{\mu_0^2}{h\gamma^2}\left[\frac{2\mu_0^2(1-\mu_0)}{h} + \gamma^2\mu_0^{1/2}(1-\mu_0)\right] + \cdots \\
\overline{p_0^3\mu} - \overline{p_0^2}\,\overline{p_0\mu} &= \frac{4}{N}\frac{\mu_0(1-\mu_0)}{h\gamma^2}\left[\frac{8\mu_0^3}{h} - \frac{4\mu_0^2}{h} + 2\gamma^2\mu_0^{1/2}(3\mu_0-1) + \gamma^4 h\right] + \cdots \\
\overline{p_0\mu^3} - \overline{p_0\mu}\,\overline{\mu^2} &= \frac{8}{N}\frac{\mu_0^3}{h\gamma^2}\left[\frac{2\mu_0}{h}(1-2\mu_0) - \gamma^2\mu_0^{1/2}\right] + \cdots
\end{aligned}
\tag{B.60}
$$

Substituting Eqs. (B.60) in (B.26) produces astonishing cancellations and we find

$$
\text{sum of first six terms in } \frac{\langle V_4^2\rangle}{8N} = \frac{16\mu_0^5}{h^2\gamma^2} + \frac{24\mu_0^{5/2}}{h} + 9\gamma^2 \tag{B.61}
$$

From (B.48), (B.44), and (B.53′), this is

$$
-\frac{16\mu_0^5}{h(2\mu_0^{3/2} + h\Lambda_2)} + \frac{24\mu_0^{5/2}}{h} - 9\left(\frac{2\mu_0^{3/2}}{h} + \Lambda_2\right)
$$

Letting $N \to \infty$ gives

$$
\begin{aligned}
&\lim_{N\to\infty}\left(\text{sum of first six terms in } \frac{\langle V_4^2\rangle}{8N}\right) \\
&\quad = -16\left(\frac{h}{2s^* - \nu g(0)}\right)^{10} \\
&\quad\quad \times \left\{h\left(2\left[\frac{h}{2s^* - \nu g(0)}\right]^3 + h\frac{1}{2\pi}\int_0^{2\pi}\frac{d\theta}{[2s^* - \nu g(\theta)]^2}\right)\right\}^{-1} \\
&\quad\quad + 24\frac{h^4}{[2s^* - \nu g(0)]^5} - 18\frac{h^2}{[2s^* - \nu g(0)]^3} \\
&\quad\quad - 9\frac{1}{2\pi}\int_0^{2\pi}\frac{d\theta}{[2s^* - \nu g(\theta)]^2}
\end{aligned}
\tag{B.62}
$$

where we have used the fact that Λ_2 [Eq. (B.45)] is a Riemann sum and s^* is determined by (B.6). Combining this result with (B.39′), we find that all the "dangerously divergent" terms miraculously disappear, leaving

$$-9\frac{1}{2\pi}\int_0^{2\pi}\frac{d\theta}{[2s^*-\nu g(\theta)]^2}$$
$$+\left[8\left(\frac{h}{2s^*-\nu g(0)}\right)^7\frac{1}{2\pi}\int_0^{2\pi}\frac{d\theta}{[2s^*-\nu g(\theta)]^2}\right]$$
$$\times\left\{2\left(\frac{h}{2s^*-\nu g(0)}\right)^3+h\frac{1}{2\pi}\int_0^{2\pi}\frac{d\theta}{[2s^*-\nu g(\theta)]^2}\right\}^{-1}$$

Letting $h\to 0$, this becomes [with s^* determined as in (8), (B.35), and (B.36)]

$$-9\frac{1}{2\pi}\int_0^{2\pi}\frac{d\theta}{[2s^*-\nu g(\theta)]^2}, \qquad \nu\leqslant\nu_c$$

$$-\frac{9}{\nu^2}\frac{1}{2\pi}\int_0^{2\pi}\frac{d\theta}{[g(0)-g(\theta)]^2}+\frac{4}{\nu^2}\left(1-\frac{\nu_c}{\nu}\right)^2\frac{1}{2\pi}\int_0^{2\pi}\frac{d\theta}{[g(0)-g(\theta)]^2}, \qquad \nu\geqslant\nu_c$$

Combining this with (B.36) and (B.37), we find that the first terms cancel by Parseval's identity, yielding (with s^* determined as above)

$$\lim_{h\to 0}\lim_{N\to\infty}\frac{\langle V_4{}^2\rangle}{8N}=\begin{cases}3\displaystyle\sum_{n=-\infty}^{\infty}\left[\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{in\theta}\,d\theta}{2s^*-\nu g(\theta)}\right]^4, & \nu\leqslant\nu_c\\[2ex] \displaystyle\frac{3}{\nu^4}\sum_{n=-\infty}^{\infty}\left[\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{in\theta}\,d\theta}{g(0)-g(\theta)}\right]^4 & \\ \displaystyle\quad+\frac{12}{\nu^3}\left(1-\frac{\nu_c}{\nu}\right)\sum_{n=-\infty}^{\infty}\left[\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{in\theta}\,d\theta}{g(0)-g(\theta)}\right]^3 & \\ \displaystyle\quad+\frac{4}{\nu^2}\left(1-\frac{\nu_c}{\nu}\right)^2\frac{1}{2\pi}\int_0^{2\pi}\frac{d\theta}{[g(0)-g(\theta)]^2}, & \nu\geqslant\nu_c\end{cases}$$

ACKNOWLEDGMENTS

The author wishes to thank Mark Kac of the Rockefeller University for suggesting the above problem and acting as his advisor throughout the investigation of it. He is grateful for his invaluable insights. He also wishes to express thanks to Jerome K. Percus of Courant Institute of Mathematical Sciences for many helpful discussions and ideas.

REFERENCES

1. T. H. Berlin and M. Kac, *Phys. Rev.* **86**:821 (1952).
2. W. W. Barrett and M. Kac, *Proc. Nat. Acad. Sci. U.S.A.* **72**(12):4723 (1975).
3. W. W. Barrett, An Intermediate Spherical Model of a Ferromagnet, Ph.D. Thesis, Courant Institute of Mathematical Sciences, New York University (1975).